

Conformation characters of gel sheets with rotational symmetry: the role of boundary

Xiaobo Zhai^{*a}, Shengli Zhang^{a,b} and Shumin Zhao^c

^a*Department of Applied Physics, Xi'an Jiaotong University, Xi'an 710049, China.*

^b*MOE Key Laboratory for Nonequilibrium Synthesis and Modulation of Condensed Matter, Xi'an Jiaotong University, Xi'an 710049, China.*

^c*Department of College Physics, Xi'an Jiaotong University, Xi'an 710049, China.*

In this paper, we systemically study the conformation characters of rotational symmetric gel sheets with free boundary and investigate the role of boundary on the equilibrium conformation. In gel sheet the boundary provides a residual strain which leads to re-distribution of stress and impacts the shape of equilibrium conformation accordingly. For sheet with boundary, the in-plane stretching energy is far larger than the bending energy in some cases. It is intrinsic different from closed membrane. In gel sheets, the boundary doesn't only quantitatively amend to the elastic energy. The residual strain on boundary cooperates with bending and stretching to determine the equilibrium conformation rather than just the last two factors. Furthermore, on the boundary of gel sheet, there is an additional energy induced by boundary line tension γ . If $\gamma = 0$, there is 10% difference of elastic energy from the experimental result. Finally, we discuss the effects of such line tension γ and propose a way to measure it by the border radius. It redounds to study the physical origination of γ .

INTRODUCTION

In thin biological tissues, the surface shapes are determined by their gene [1 – 8] and external force [9 – 11]. The sheets with boundary have very different shape from the closed membrane which has no boundary. For example, the Gaussian curvature on boundary is negative, such as the leaves of *Acetabularia schenckii* [5]; the vesicle with holes [12]; the leaves with wavy edges which like torn edge of plastic garbage bags [1]. These phenomena are general and can be found in other membrane such as the synthetic gel sheets [13]. The rotational symmetric gel sheets have brim-like boundary [13, 14], while the non-rotational symmetric gel sheets are wavy-like [13 – 16]. These sheets also have negative Gaussian curvature on boundary.

Theoretical studies reveal that the boundary of gel sheets have negative Gaussian curvature [17, 18]. The numerical simulations demonstrate that the boundary regions of leaves with wavy

* xzbzhai8326@163.com

shape and vesicle with positive Poisson ratio all have negative Gaussian curvature near edges [19, 20], which support our previous theoretical conclusion [18]. The similar results are also found for the sheets with elliptic metric by E. Sharon *et al.*, who point out the boundary layers have abundant stretching energy [21]. These discoveries imply that the sheet with free boundary has special geometric shape and may have particular mechanical character, especially on the range near its boundary.

On the boundary, because the molecules are difference between two sides of boundary, there has a residual force. Then, boundary will have an additional energy which can be described by a line energy density as the boundary line tension γ [24, 25]. In many researches, people generally regard that the changing of elastic energy arisen from γ is weak. Therefore, in theoretical studies γ usually be omitted [6, 17, 21 – 23].

In this paper, we investigate the role of boundary on reforming the equilibrium shape and the distribution of strain for rotational symmetric gel sheets, and study the effect of boundary line tension γ on the equilibrium conformation. Firstly, the boundary has a residual strain which arises from the deformation to meet the boundary condition. This strain leads to the geometric and mechanic characters in a sheet with boundary obvious different from the sheet without boundary. Because of the residual strain, the in-plane energy is enlarged near boundary. Then, the elastic energy in a gel sheet with boundary is even two times larger than that in the sheet without boundary. Consequently, the equilibrium conformation is determined by the competition of bending, stretching and residual strain rather than just the first two factors. Thus, in gel sheets, the boundary doesn't just quantitatively amend to the elastic energy, but has essential influence on the elastic energy. Secondly, γ will decrease the border radius and elastic energy of gel sheet generally. If leaving out the boundary line tension, the elastic energy has almost 10% difference from the experimental result. Thus, in studies of gel sheets, people need to consider the boundary line tension. Finally, by the relationship between γ and equilibrium conformation, we propose a simple method to measure boundary line tension by the border radius.

This paper is organized as follows: In Sec. II, the equilibrium shape equations of gel sheets are deduced. It is found that the boundary has special Gaussian curvature which is determined by Poisson ratio (Eqs. (6)). In Sec. III, we study the physical effects of boundary. In closed membrane, the stretching will reduce the conformation energy. However, in some cases, the stretching increases the conformation energy because of the accumulation of stretching energy near boundary. The stretching energy is even larger than the bending energy. Then, we compare the sheet with and without boundary and find the boundary has tremendous effect to the equilibrium conformation. In

Sec. IV, we research the relationship between the equilibrium conformation and the boundary line tension. The boundary line tension and the border radius have monotonic decreases relationship. Section V is a conclusion.

I. THE EQUILIBRIUM SHAPE EQUATIONS

In the gel sheets, we define two surface states, named as target state \tilde{S} and equilibrium state S . The target state \tilde{S} is a state which is in-plane strain free and it is an ideal conformation. The equilibrium state S is the final stable conformation of gel sheets.

The thin gel sheets can be represented as a 2D curved surface. For thin sheets, the conformation energy can be written as $E = E_s + E_b + E_C$ where E_s is in-plane stretching energy, E_b is bending energy and E_C is an additional energy from the boundary line tension. The elastic energy F in the gel sheet is noted as $E_s + E_b$. The in-plane stretching energy E_s is represented by the displacement 2D vector field \mathbf{u} on the surface of sheet which is determined by the difference between equilibrium state S and the target state \tilde{S} [18]. The bending energy E_b is denoted as Helfrich [26]. The conformation energy is [26, 27]

$$\begin{aligned} E &= \frac{1}{2} \int_{\tilde{S}} w_s d\tilde{A} + \int_S w_b dA + \oint_C \gamma ds, \\ w_s &= 2\mu u_{\alpha\beta}^2 + \lambda u_{\alpha\alpha}^2, \\ w_b &= \frac{\kappa}{2} H^2 + \kappa_G K, \end{aligned} \tag{1}$$

where w_s and w_b are the energy densities of in-plane stretching energy and bending energy, $d\tilde{A}$ and dA are the area element on the surface \tilde{S} and S , respectively. H is the mean curvature (we adopt it as the sum of principal curvatures rather than the average). K is the Gaussian curvature.

For 2D surface, the elastic coefficients μ , λ , κ and κ_G satisfy [28]

$$\begin{aligned} 2\mu + \lambda &= \frac{hY_0}{1 - \hat{\nu}^2}, \\ \lambda/(2\mu + \lambda) &= \hat{\nu}, \\ \kappa/(2\mu + \lambda) &= \frac{h^2}{12}, \\ \kappa_G/\kappa &= \hat{\nu} - 1, \end{aligned} \tag{2}$$

where h is the thickness of sheet, Y_0 is Young modulus and $\hat{\nu}$ is the Poisson ratio.

For the sheet with a boundary C , the amount and type of molecules between two sides of its boundary are different (gel's molecule is inside, and environmental molecule is out side), so the

boundary suffers a residuals force which induce an additional energy on boundary. We can define a boundary line tension γ , which represents the line density of additional energy on boundary. This additional energy can be written as $E_C = \oint_C \gamma ds$ [24, 25]. In gel sheet, the shape of target state is determined by the initial concentration, so the value of $\oint_C \gamma d\tilde{s}$ is constant. It can be regarded as a referenced potential energy. Then, the boundary term of gel sheets is

$$E_C = \oint_C \gamma ds - \oint_C \gamma d\tilde{s}. \quad (3)$$

where ds and $d\tilde{s}$ are the line element on the boundary of equilibrium state and target state, respectively.

For the surface \tilde{S} and S of rotational symmetric sheet, we choose cylindrical coordinates $(\tilde{\rho}, \varphi, \tilde{z}(\tilde{\rho}, \varphi))$ and $(\rho, \varphi, z(\rho, \varphi))$ to describe them, respectively. The in-plane strain tensor \mathbf{u} is [18, 29]

$$\begin{aligned} u_{11} &= \frac{1}{2} \left[\left(\frac{\rho}{\tilde{\rho}} \right)^2 - 1 \right], \\ u_{22} &= \frac{1}{2} \left[\left(\frac{dl}{d\tilde{l}} \right)^2 - 1 \right], \\ u_{12} &= u_{21} = 0. \end{aligned} \quad (4)$$

where $\tilde{l} = \int \sqrt{1 + \tilde{z}_{\tilde{\rho}}^2} d\tilde{\rho}$ and $l = \int \sqrt{1 + z_{\rho}^2} d\rho$. If we denote $Y = -\frac{z_{\rho}}{\sqrt{1+z_{\rho}^2}}$, it has

$$\begin{aligned} a &= \frac{Y}{\rho}, \\ c &= \frac{dY}{d\rho}. \end{aligned} \quad (5)$$

Substituting Eqs. (5) and Gauss-Bonnet formula $\int \int K dA = 2\pi - \oint_C k_g ds$ [30] into Eqs. (1), and using the variational principle $\delta F + \delta E_C = 0$, we can obtain the equilibrium equations of rotational symmetric sheets

$$\begin{aligned} M_1 - \nabla_1(M_2 k_g^{-1}) + \kappa \left\{ \frac{1}{2} (H^2 - 4k_n^2) + \frac{1}{2} \nabla_1[(H^2 - 4K)k_g^{-1}] \right\} &= 0, \\ aM_2 + \kappa \left[\frac{1}{2} aH^2 - \frac{F^4}{\rho^2} \nabla_1(Hk_g^{-1}) + H \frac{F^4}{\rho^2} \right] &= 0, \end{aligned} \quad (6)$$

and the sheets must satisfy the boundary conditions

$$M_2 + \frac{\kappa}{2}H(a - c) + \gamma k_g = 0, \quad (7)$$

$$c + \hat{\nu}a = 0.$$

where $F = \sqrt{1 + z_\rho^2}$, $\nabla_1 = \frac{d}{dl}$, $M_1 = \sigma_{11} \frac{\rho}{\tilde{\rho}} \frac{d\tilde{l}}{dl}$, $M_2 = \sigma_{22} \frac{\tilde{\rho}}{\rho} \frac{dl}{d\tilde{l}}$, $k_g = \frac{1}{\rho F}$, $k_n = a$. $\sigma_{11} = (2\mu + \lambda)(u_{11} + \hat{\nu}u_{22})$, $\sigma_{22} = (2\mu + \lambda)(u_{22} + \hat{\nu}u_{11})$. σ_{11} and σ_{22} are the stress along circumferential direction and radial direction, respectively. The second equation of Eqs. (7) decides the relationship between $\hat{\nu}$ and the sign of c/a , which also exists for non-rotational symmetric sheets [18, 24].

We use reduction method to obtain the equilibrium equations (6). Comparing with the previous work [18], these equations are easily to be solved and need lesser boundary condition. The second equation of Eqs. (7) is universal in the studies of equilibrium shape. This equation will not be changed by different variation method [18, 24]. We note that the second equation of Eqs. (7) determines the Gaussian curvature of boundary.

The conformations of gel sheets are determined by the equilibrium equations (6) and boundary conditions (7) together. On the boundary, the sheets must obey boundary conditions Eqs. (7) and need satisfy the equilibrium Eqs. (6). It implies that the boundary also governs the equilibrium conformation except for the bending and in-plane stretching. This is different from closed membrane in which the equilibrium conformation is just determined by the competition of bending and in-plane stretching. In this paper, we research the open gel sheets with free boundary and discuss the effects of boundary by comparing the sheets with and without boundary.

II. CONFORMATION CHARACTERS INDUCED BY BOUNDARY

In biologic tissues, researchers observe that some thin rotational symmetric membranes have negative Gaussian curvature on the boundary although the Gaussian curvature is positive in the central regions, such as the leaves of *Acetabularia schenckii* [5] and the vesicle with holes [12]. The NIPA gel sheets of dome-like shape also have negative Gaussian curvature on the boundary [13]. The gel sheets which simulate growth of leaves have gradual changed initial concentration [13]. Similarly as the growing process of a leave, the shrinkage ratio in the artificial gel sheet can be regarded as the growth. One can use shrinkage ratio η_0 , which is linear with initial concentration, to describe the target states as Ref. 13 and Ref. 18.

We classify the sheets by their target state's shape. The rotational symmetric gel sheets have two types. In their target state, one has positive Gaussian curvature and another has negative

Gaussian curvature, such as dome-like sheets and torus-like sheets. From the experiment [13], we know that the shape of target state is determined by η_0 . The shrinkage ratio η_0 of dome-like and torus-like sheets are $\sim r^n$ and $\sim r^{-n}$, respectively ($n > 0$). To demonstrate the character of Gaussian curvature on the boundary, we will investigate these two types of gel sheets (if with no special annotation, the radius and thickness of sheet are $r_{\max} = 5\text{cm}$ and $h = 0.5\text{mm}$). Then, from these different sheets, the boundary characters are summarized by the calculated data.

Furthermore, in order to show the conformation characters induced by the boundary, for the gel sheet we compare its conformation in the equilibrium state with boundary and without boundary, respectively. To deal with the equilibrium state without boundary, we discuss an ideal infinite sheet and calculate the equilibrium equations from inside radius and cut off at the corresponding boundary r_{\max} (we doesn't attend the part beyond the boundary at r_{\max}).

In this section, we ignore the boundary line tension and will discuss it detailedly in next section.

A. Gaussian curvature

From the boundary condition Eqs. (7), we derive that the surface must satisfy $c/a = -1 - \kappa_G/\kappa = -\hat{\nu}$ on the boundary. In rotational symmetric sheet, it has $b = 0$. The Gaussian curvature is $K = ac - b^2$, so the value of Gaussian curvature on boundary ($K|_C$) is determined by the Poisson ratio $\hat{\nu}$. It can explain why on boundary the sheets have negative Gaussian curvature in the experimental cases. If $\hat{\nu} > 0$, on the boundary a and c have the opposite sign, so the $K|_C$ is negative. If $\hat{\nu} = 0$, on the boundary Gaussian curvature is zero. And if $\hat{\nu} < 0$, the boundary has positive Gaussian curvature. Generally the NIPA gel sheets have positive Poisson ratio [31], so the boundary of rotational symmetric gel sheets must has negative Gaussian curvature.

Firstly, we study the dome-like sheets. To an experimental sheet (a dome-like sheet in the Fig. 2b of Ref. 13), we found that it really has negative $K|_C$ by both the experimental data [13] and our theoretical calculation (Table 1). From the boundary condition Eqs. (7), we derive that the $K|_C$ are determined by the sign of Poisson ratio as Fig. 1a. For dome-like sheets with different target shape, in equilibrium state the Gaussian curvature of them have similar distribution (Fig. 1). The increase of initial concentration gradient just enlarges the extent of Gaussian curvature (Fig. 1b). Furthermore, for different initial size and thickness (we consider the thin thickness when the sheets can be bending), the boundary also has the special geometric character: the absolute value of $K|_C$ is enhanced by the initial size increasing or the thickness decreasing (Fig. 2). The distribution of Gaussian curvature is different in equilibrium state and target state (Fig. 1a). For

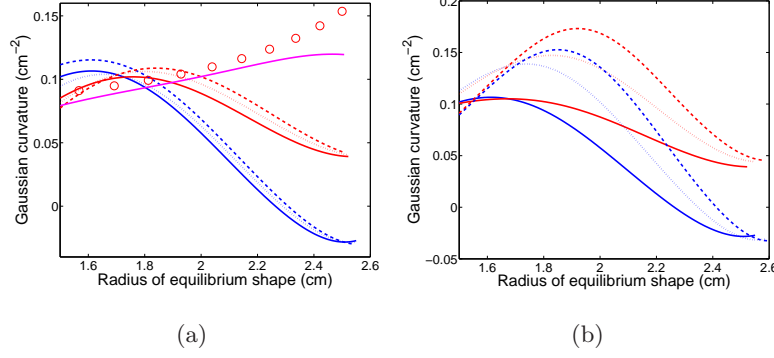


Fig. 1: (a) The distribution of Gaussian curvature. The gel sheet has $\eta_0 = 0.6 - r^2/250$. The hollow-dot line is in target state. The pink line is in the sheet without boundary when $\hat{\nu} = 0.5$. The real line is in equilibrium state when $\hat{\nu} = 0.5$ (blue line) and $\hat{\nu} = -0.5$ (red line) to different boundary line tension, such as $\gamma/(2\mu + \lambda) = 0\text{cm}$ (real line), 0.0005cm (dot-dash line), 0.001cm (dashed). (b) The distribution of Gaussian curvature in different sheet. $\eta_0 = 0.6 - r^2/250$ (real line), $\eta_0 = 0.6 - r^3/1250$ (dot-dash line) and $\eta_0 = 0.6 - r^4/6250$ (dashed) when $\hat{\nu} = 0.5$ (blue line) and $\hat{\nu} = -0.5$ (red line).

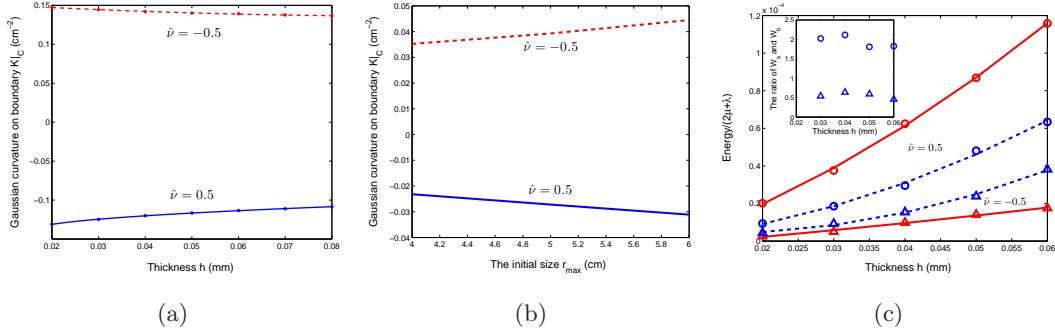


Fig. 2: (a) The thickness vs. Gaussian curvature on boundary $K|_C$. (b) The initial size r_{\max} vs. $K|_C$. (c) The in-plane stretching energy (red line) and bending energy (blue line) increase with thickness (the dot line with $\hat{\nu} = 0.5$ and the triangle line with $\hat{\nu} = -0.5$). The initial shrinkage ratio of gel sheet is $\eta_0 = 0.6 - r^2/250$.

dome-like sheets, the target states have positive Gaussian curvature. When $\hat{\nu} > 0$, near boundary the Gaussian curvature of equilibrium states is negative. When $\hat{\nu} < 0$, the boundary of equilibrium state has positive Gaussian curvature. For torus-like sheets, the Gaussian curvature of target state is negative. When $\hat{\nu} > 0$ the boundary of equilibrium state also has negative Gaussian curvature. But when $\hat{\nu} < 0$, the $K|_C$ is positive. For both dome-like sheet and torus-like sheet, the $K|_C$ decrease when the Poisson ratio increases (Fig. 3). And Gaussian curvature on the boundary depends on the Poisson ratio as $K|_C \sim \hat{\nu}^2$.

Because the Poisson ratio determines the value of c/a , it restricts the shape near boundary. To obtain the sheets with special surface, such as sphere and cylinder, we must use special material

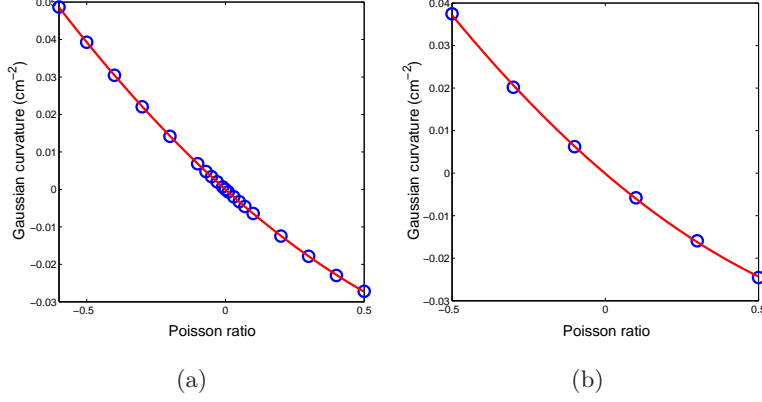


Fig. 3: The Gaussian curvature on the boundary vs. Poisson ratio. (a) The dome-like sheet with $\eta_0 = 0.6 - r^2/250$ and (b) the torus-like sheet with $\eta_0 = 0.4 + 1/2r^2$. The inset is elastic energy vs. Poisson ratio.

which has the designated Poisson ratio. For the half-sphere and cylinder sheets, they need $\hat{\nu} = 0$ which derived from the boundary condition (7). However, in gel sheets the Poisson ratio is not zero generally, so the cylinder sheets made by experiment must have an additional edge in the experiment [13].

B. Strain

Between the target state and equilibrium state, the sheets have different Gaussian curvature especially near boundary. So, the equilibrium state has different shape from the target state. One may expect that the sheet has a deformation near boundary. Because the target state doesn't have in-plane stretching, this deformation will induce a residual strain near boundary.

Using Eqs. (4), we obtain the strain/stress on the equilibrium shape and obtain the density of stretching/bending energy (Fig. 4). For dome-like sheets, the bending energy mainly distribute in the center region, contrarily the stretching energy almost distributes near the boundary (Fig. 4a and Fig. 5a). On the center region the bending energy is greater ten times than the stretching energy, but near boundary the bending energy is much smaller than the stretching energy. The stress mainly locates on the circumferential direction, and is nearly zero along radial direction. When $\hat{\nu} > 0$, along two direction (radial and circumferential) the strains have different sign, but when $\hat{\nu} < 0$ these two strains have same sign. For torus-like sheets, when $\hat{\nu} > 0$ the $K|_C$ has same sign between target and equilibrium states, so there just has a small strain near boundary (Fig. 5b). In this case, though the in-plane stretching energy is smaller than the bending energy, the additional stretching energy on boundary also exist. And the additional stretching energy becomes

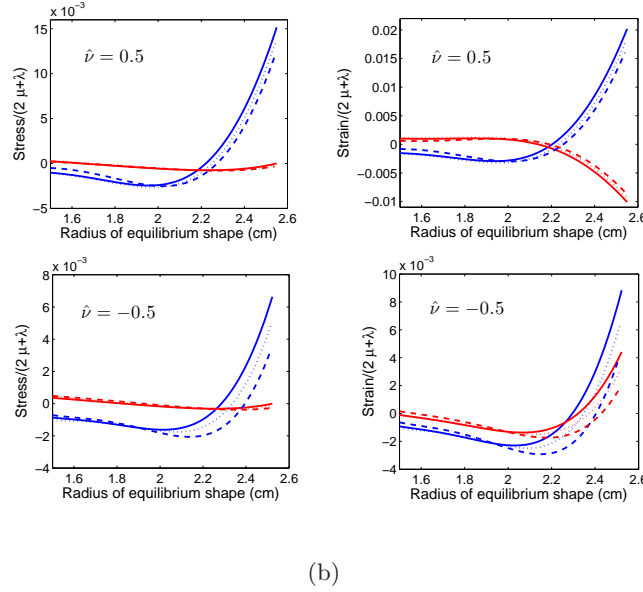
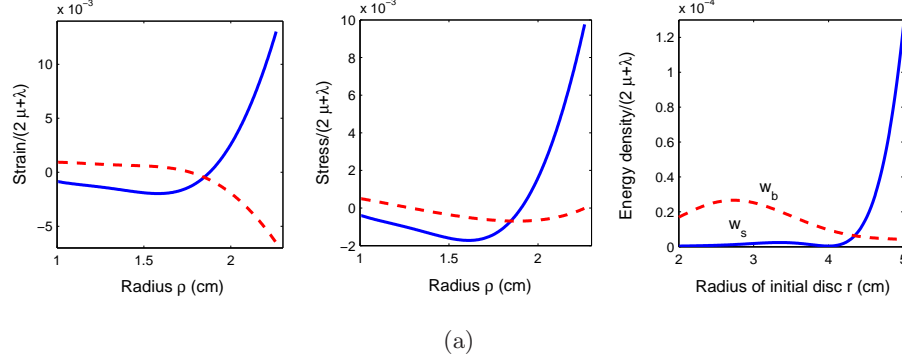


Fig. 4: (a) The elastic character of the gel sheet in experiment. The first two figures are the strain and stress distribution of equilibrium shape along radial direction (red line) and circumferential direction (blue line). The third figure is the distribution of bending energy (red line) and stretching energy (blue line), respectively. In the central region, the bending energy has a maximum value at $r = 2.75\text{cm}$ and the stretching energy has a maximum value at $r = 3.4\text{cm}$. Near the boundary, the stretching energy augment rapidly. (b) The strain and stress distribution of equilibrium shape to different boundary line tension, such as $\gamma/(2\mu + \lambda) = 0\text{ cm}$ (real line), 0.0005cm (dot line), 0.001cm (dashed). The strain and stress along radial direction (red line) and circumferential direction (blue line) for the gel sheet with $\eta_0 = 0.6 - r^2/250$.

larger when $\hat{\nu} < 0$.

In the experimental sheet, the border radius of equilibrium state and target state are 2.262cm and 2.233cm when $\hat{\nu} = 0.5$, respectively. And in the sample D1 (we mark that the sheets with $\eta_0 = 0.6 - r^2/250$ and $\eta_0 = 0.4 + 1/2r^2$ are sample D1 and sample T1, respectively), these value are 2.571cm and 2.500cm , respectively. The border radius of equilibrium state is bigger than the one of target state. To form the negative $K|_C$, the boundary conditions make the sheet to bend outward

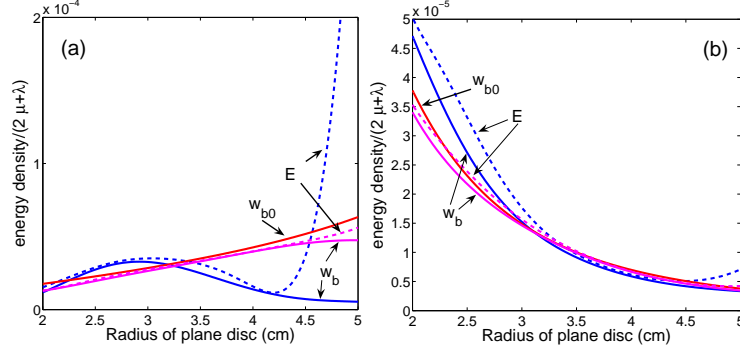


Fig. 5: The distribution of the elastic energy, the bending energy in equilibrium state (w_b) and in target state (w_{b0}) with $\hat{\nu} = 0.5$ for (a) the dome-like sheet with $\eta_0 = 0.6 - r^2/250$ and (b) the minimal surface sheet with $\eta_0 = 0.4 + 1/2r^2$. The energy distributions with boundary are blue, and without boundary are pink.

near boundary. In torus-like sheet (sample T1) with $\hat{\nu} = -0.5$, the border radius of equilibrium state and target state is 2.081cm and 2.100 cm, respectively. Then, the positive $K|_C$ needs the sheet bending inward near boundary. The border radius of equilibrium state is different from the one of target state. Therefore, there is stretching on the boundary consequentially. This stretching is reserved and gradually reduces from boundary to center (Fig. 4). Obviously, to attain the special shape on boundary, the sheets will acquire a deformation and have residual strain which induces the additional stretching energy.

The ratio of bending and stretching energy is determined by Poisson ratio (Fig. 6). In dome-like sheets, the stretching energy is larger than the bending energy for general gels ($0.3 < \hat{\nu} < 0.5$). However, for the material with $\hat{\nu} < -0.2$, in despite of the stress also cumulates near boundary, the stretching energy change to smaller than the bending energy, and almost disappear when $\hat{\nu} = -0.7$. There is a critical value $\hat{\nu}_C$ about the transition of domination between stretching and bending energy. The $\hat{\nu}_C$ is near -0.2 which is close to the midpoint on the extent of Poisson ratio ($-1 \leq \hat{\nu} \leq 0.5$). With large Poisson ratio, the stretching energy is dominant. But, with small Poisson ratio, the bending energy is dominant. In torus-like sheets, the stretching energy is smaller than the bending energy. And the stretching energy nearly disappear when $\hat{\nu} = 0.5$. With large positive $\hat{\nu}$ in dome-like sheets and small negative $\hat{\nu}$ in torus-like sheets, the stretching energy occupies a large portion of elastic energy obviously. In these cases, the significance of boundary is very notable.

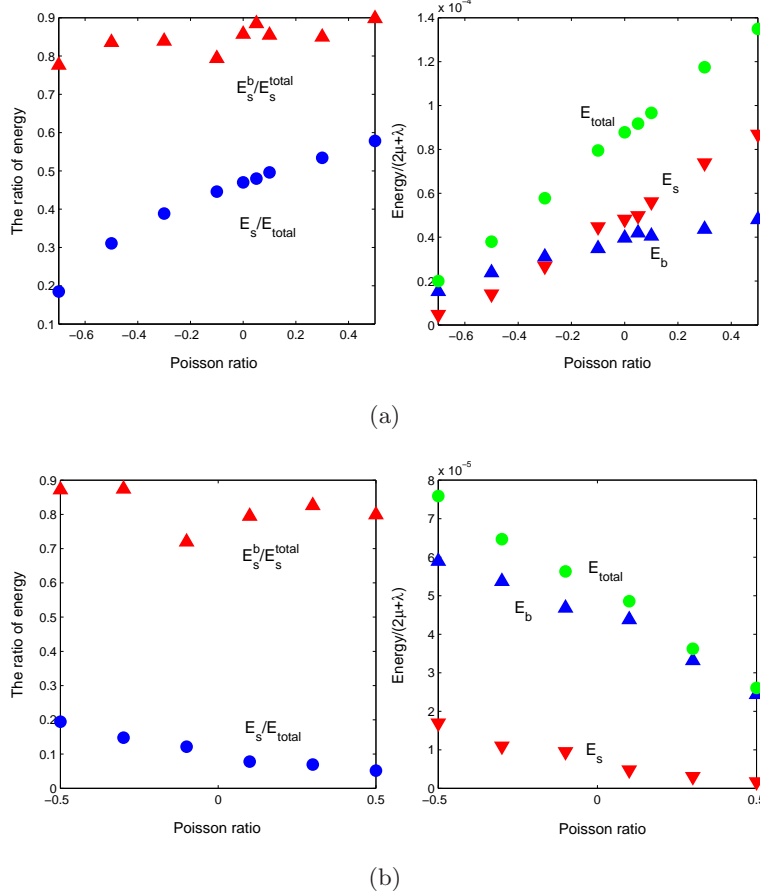


Fig. 6: The stretching energy (E_s) and bending energy (E_b) vs. Poisson ratio. The gel sheets with (a) $\eta_0 = 0.6 - r^2/250$ and (b) $\eta_0 = 0.4 + 1/2r^2$. The E_s^b is the stretching energy near boundary (the outer 10% of initial sheet). The E_s^{total} and E^{total} are the total stretching energy and total elastic energy.

C. The effect of boundary

In fact, the boundary plays a decisive role to the equilibrium shape. To discuss the role of boundary in equilibrium conformation, we compare the sheet with and without boundary. The equilibrium shape without boundary is different from the one with boundary generally. At cut off radius r_{max} (which corresponding to the border radius of the sheet with boundary), we label the Gaussian curvature with and without boundary as $K|_C^b$ and $K|_C^{nb}$, respectively. Comparing $K|_C^b$ and $K|_C^{nb}$, we know that the boundary will change the value of Gaussian curvature. In dome-like sheets, with $\hat{\nu} < 0$ the $K|_C^b < K|_C^{nb}$. And with $\hat{\nu} > 0$, the $K|_C^b$ and $K|_C^{nb}$ even have opposite sign. In torus-like sheets, with $\hat{\nu} > 0$ the $K|_C^b > K|_C^{nb}$, and they have opposite sign with $\hat{\nu} < 0$ (Table 1). Furthermore, without boundary the distribution of Gaussian curvature on equilibrium state just has a little different from that on target state, however this difference is obvious in the sheet with

TABLE I: The Gaussian curvature on boundary $K|_C$ and elastic energy with and without boundary (the data with asterisk are with boundary, for without boundary sheets we use the data which cut off at r_{max}) in experimental sheet, dome-like sheet with $\eta_0 = 0.6 - r^2/250$ (sample D1), and torus-like sheet with $\eta_0 = 0.4 + 1/2r^2$ (sample T1). The change rate ϵ is the difference of elastic energy between equilibrium state and target state (in target state the elastic energy just contains bending energy because the target state has no in-plane strain).

	$K _C$ (cm ⁻²)	$E/(2\mu + \lambda)$	ϵ (%)
experimental sheet	0.090	6.83	-7.5
($\hat{\nu} = 0.5$)	-0.027*	8.18*	10.8*
sample D1	0.120	10.10	-8.4
($\hat{\nu} = 0.5$)	-0.22*	25.58*	132.2
sample D1	0.113	3.73	-2.5
($\hat{\nu} = -0.5$)	0.039*	4.19*	9.7
sample T1	-0.036	3.89	-2.0
($\hat{\nu} = 0.5$)	-0.025*	4.89*	23.2
sample T1	-0.077	11.55	-3.1
($\hat{\nu} = -0.5$)	0.37*	14.37*	20.7

boundary especially near boundary (Fig. 3a).

Furthermore, a sheet with boundary has larger elastic energy than the sheet without boundary (Table 1). In both dome-like and torus-like sheets, the boundary makes the elastic energy augment whether $\hat{\nu} > 0$ or $\hat{\nu} < 0$. Especially, in dome-like sheet when $\hat{\nu} > 0$ and torus-like sheet when $\hat{\nu} < 0$, the augment of elastic energy is notable. In dome-like sheet this augment mainly gather near boundary and is very larger than that in torus-like sheet which doesn't have obvious augment near boundary (Fig. 5). And, the increase of elastic energy is two times more than that in the torus-like sheet. So, when the deformation on boundary is notable (the sign of $K|_C$ between equilibrium state and target state are opposite), the boundary will increase elastic energy obviously.

In the sheet both with and without boundary, comparing to the target state, the stretching will reduce the bending energy. But, in the sheet with boundary, the decrease of bending energy is more obvious. Without boundary, the stretching makes the elastic energy of equilibrium state to be smaller than that of the target state. However, considering the boundary, the equilibrium state has larger elastic energy than the target state (Table 1). The boundary is important and has a great contribution to the elastic energy. Thus, we can find that the boundary plays an important

role to format the shape and re-distribute the stress in equilibrium conformation.

In a word, the boundary has very important effect to the strain. There is a lot of strain accumulated near boundary. The 10% of most outer part of initial sheet has 90% of total in-plane stretching energy (Fig. 6). These boundary characters can be observed in the gel sheets with different distribution of concentration and with either positive or negative Poisson ratio. And with different initial size and thickness, the boundary always induces the special characters of equilibrium shape and strain distribution. The residual strain even makes the in-plane stretching energy to be larger than the bending energy in some cases. Without boundary, the stretching will decrease the elastic energy as close membranes. However, with boundary, the stretching makes the elastic energy increase. Thus, the boundary has a decisive status to determine the elastic energy. In theoretical studies people cannot ignore the boundary.

III. CHARACTERS OF BOUNDARY LINE TENSION

The theoretical discussion of equilibrium shape didn't consider the boundary line tension in the previous studies [17, 18, 21]. However, the boundary term E_C is an important component of conformation energy [24, 27]. In this section, we will consider the boundary line tension and discuss its effect.

A. Border radius

Using the equilibrium shape equations (Eqs. (6)), we obtain the equilibrium shape with different boundary line tension. In dome-like sheets, the boundary line tension and the border radius has a linear relationship for both positive and negative Poisson ratio (Fig. 7). The similar linear relationship also can be found in the torus-like sheets. The border radius of equilibrium shape will decrease when the boundary line tension augment. So the boundary line tension will promotes the shrinkage on the edge. The border radius is easily to be measured in the experiment. Its theoretic value can be compared with the experimental data in straightway. In experiment, the accurate boundary line tension is measured hardly. Fortunately, the linear relationship between boundary line tension and border radius can offers a method to measure the boundary line tension by the border radius which can be obtained directly.

From the experiment observation [13], the border radius of gel sheet is 2.2584cm. However, it is 2.262cm from the theoretical calculation with no boundary line tension [18]. Apparently, the

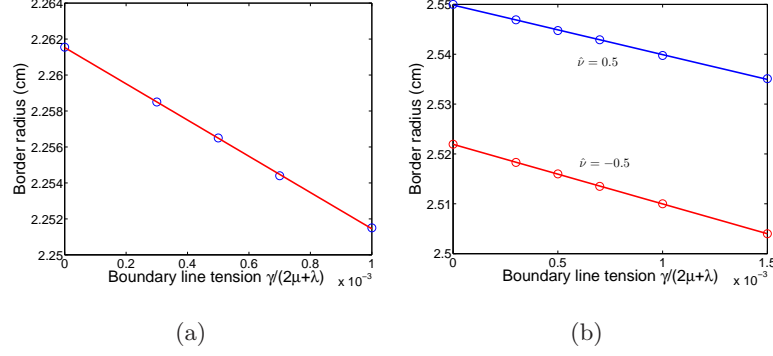


FIG. 7: (a) Boundary line tension vs. border radius for the experimental sheet. (b) The relation between boundary line tension and border radius for $\hat{\nu} = 0.5$ (blue line) and $\hat{\nu} = -0.5$ (red line) in the gel sheets with $\eta_0 = 0.6 - r^2/250$.

theoretical border radius with no boundary line tension is bigger than the experimental observation data. For precise calculation, the boundary line tension is needed. The theoretical border radius is same as the experimental data (2.2584cm) in case of $\gamma/(2\mu+\lambda) = 0.00031\text{cm}$ (Fig. 7a). Considering the Young modulus of NIPA gel is 0.11MPa [30] and from Eqs. (2), it implies that the corresponding boundary line tension is $2.3 \times 10^{-5}\text{J/cm}$. Furthermore, the residual strains on boundary with and without γ are 0.0132 and 0.0116, respectively. We know that the residual strain decides the equilibrium conformation of gel sheet. The elastic energy $F/(2\mu+\lambda) = 7.547 \times 10^{-5}$ and 8.180×10^{-5} for with and without γ , respectively. So, although the difference of border radius between with and without γ is small, the γ also has a big influence to the equilibrium conformation.

B. Strain

The different boundary line tension just affects the magnitude of Gaussian curvature and does not change the sign of K on boundary (Fig. 1a). In the dome-like sheets, when $\hat{\nu} > 0$ the strain and stress almost distribute near the boundary (Fig. 4). For small boundary line tension, it will weaken the in-plane stretching but just have less influence to the distribution of strain and stress (Fig. 4b).

The bending energy weakly enhances with boundary line tension γ (Fig. 8). For rotational symmetric sheets, the boundary term of conformation energy is $E_C = \oint_C \gamma ds = 2\pi\gamma R_C$. With large boundary line tension γ , the border radius R_C will tend smaller to reduce the energy of E_C . Along the boundary, the displacement is $R_C - \tilde{R}_C$, so the stretching energy is proportion to $|R_C - \tilde{R}_C|$. And because the γ will decrease R_C , the large γ weakens the stretching energy when

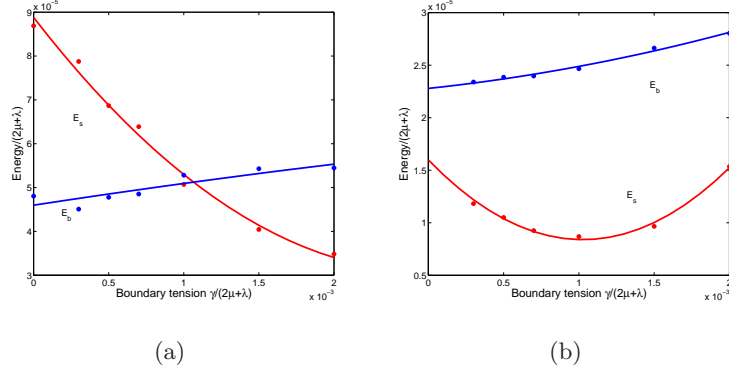


Fig. 8: The boundary line tension vs. the stretching energy E_s (red line) and bending energy E_b (blue line). The gel sheets with $\eta_0 = 0.6 - r^2/250$ when (a) $\hat{\nu} = 0.5$ and (b) $\hat{\nu} = -0.5$.

$R_C > \tilde{R}_C$. But when R_C becomes smaller than \tilde{R}_C the γ will make the stretching energy to increase. In dome-like sheets, when $\hat{\nu} > 0$ the boundary need to spread outward. The bending energy is increase with γ increasing. But the in-plane stretching energy is decrease with γ increasing. There is a critical point at $\gamma/(2\mu + \lambda) = 0.001$ in which point the domination between stretching and bending energy is transition. When $\hat{\nu} < 0$, with γ increasing the in-plane stretching energy decrease firstly and increase when $\gamma/(2\mu + \lambda) > 0.001$ (Fig. 8). In the torus-like sheet, with $\gamma = 0$ the R_C is smaller than \tilde{R}_C for both $\hat{\nu} > 0$ and $\hat{\nu} < 0$. Thus, the in-plane stretching energy will increase with γ increasing.

In a word, the boundary line tension has obvious effect on the elastic energy. So, in exact studies of gel sheets, γ need be considered. But, γ doesn't change the type of equilibrium shape and just affects the value of elastic energy and the size of equilibrium shape. Thus, the special mechanical and geometrical characteristic on boundary is not caused by boundary line tension. In other word, the boundary characters are indeed induced by the residual strain on boundary.

IV. CONCLUSION

In this paper, we investigate the role of boundary on equilibrium conformation and study the effect of boundary line tension γ . We find that the boundary can not be ignored in gel sheets. The boundary affects the competition of bending and stretching in the equilibrium conformation of gel sheets. For the rotational symmetric gel sheets, the boundary has special Gaussian curvature which is opposite to $\hat{\nu}$. This induces a deformation and residual strain near boundary. Consequently, the 10% of most outer part of initial sheet has almost 90% of total in-plane stretching energy. In some cases, the residual strain even makes the in-plane stretching energy larger than the bending

energy.

The boundary plays an essential role in the equilibrium conformation. With or without boundary, the equilibrium shapes are different obviously. The distribution of Gaussian curvature is different in both the situations. Furthermore, without boundary, the stretching will decrease the elastic energy as closed membranes. However, the boundary will increase the elastic energy. With boundary the elastic energy of equilibrium state is larger than that of target state in some cases. These phenomena are resulted from the residual strain on boundary.

To agree well with experimental result, in theoretical studies we find that the boundary line tension γ must be involved. γ impacts the residual strain on boundary. Then, considering γ or not, the elastic energies have difference about 10%. Generally, the γ decreases the elastic energy. The boundary line tension and border radius are one-to-one correspondence, therefore there is a simple way to measure the boundary line tension by the border radius.

acknowledgement

We thank M. G. Xia, E. H. Zhang, and D. S. Lei for helpful discussions. This work was supported by the Cultivation Fund of the Key Scientific and Technical Innovation Project, Ministry of Education of China (NO. 708082) and Chinese NSF grant NO. 11074196.

-
- [1] E. Sharon, M. Marder and H. Swinney, *Am. Sci.* **92**, 254 (2004).
 - [2] J. Palatnik *et al.*, *Nature* **20**, 1 (2003).
 - [3] B. Crawford, U. Nath, R. Carpenter, and E. Coen, *Plant Physiol.* **135**, 244 (2004).
 - [4] E. Sharon, B. Roman and H. Swinney, *Phys. Rev. E* **75**, 046211 (2007).
 - [5] J. Dervaux and M. Amar, *Phys. Rev. Lett.* **101**, 068101 (2008).
 - [6] H. Liang and L. Mahadevan, *Proc. Natl. Acad. Sci. USA*, **106** 22049 (2009).
 - [7] S. Nechaev and R. Voituriez, *J. Phys. A: Math. Gen.* **34**, 11069 (2001).
 - [8] U. Nath, B. Crawford, R. Carpenter and E. Coen, *Science* **299**, 1404 (2003).
 - [9] E. Cerda and L. Mahadevan, *Phys. Rev. Lett.* **90**, 074302 (2003).
 - [10] Y. Forterre, J. Skotheim, J. Dumais and L. Mahadevan, *Nature* **433**, 421 (2005).
 - [11] D. Holmes and A. Crosby, *Adv. Mater.* **19**, 3589 (2007).
 - [12] A. Saitoh, K. Takiguchi, Y. Tanaka and H. Hotani, *Proc. Natl. Acad. Sci. USA* **95**, 1026 (1998).
 - [13] Y. Klein, E. Efrati, and E. Sharon, *Science* **315**, 1116 (2007).
 - [14] J. Kim, J. Hanna, M. Byun, C. Santangelo and R. Hayward, *Science* **335**, 1201 (2012).
 - [15] A. Boudaoud and S. Chaïeb, *Phys. Rev. E* **64**, 050601 (2001).

- [16] T. Moraa and A. Boudaoud, Eur. Phys. J. E **20**, 119 (2006).
- [17] E. Efrati, E. Sharon, and R. Kupferman, J. Mech. Phys. Solids **57**, 762 (2009).
- [18] X. B. Zhai, S. L. Zhang, L. Zhang and S. M. Zhao, Int. J. Mod. Phys. B **26**, 1250084 (2012).
- [19] Z. S. Liu, W. Hong, Z. G. Suo, S. Swaddiwudhipong and Y. W. Zhang, Comput. Mater. Sci. **49**, 60 (2010).
- [20] T. Umeda, Y. Suezaki, K. Takiguchi and H. Hotani, Phys. Rev. E **71**, 011913 (2005).
- [21] E. Efrati and E. Sharon, Phys. Rev. E **80**, 016602 (2009).
- [22] C. D. Santangelo, Europhys. Lett. **86**, 34003 (2009).
- [23] T. Mora and A. Boudaoud, Eur. Phys. J. E **20**, 119 (2006).
- [24] Z. C. Tu and Z. C. Ou-Yang, J. Phys. A **37**, 11407 (2004).
- [25] F. Jülicher and R. Lipowsky, Phys. Rev. E **53**, 2670 (1996).
- [26] W. Helfrich, Z. Naturforsch. C **28**, 693 (1973).
- [27] Z. C. Tu and Z. C. Ou-Yang, Phys. Rev. E **68**, 061915 (2003).
- [28] T. Witten, Rev. Mod. Phys. **79**, 643 (2007).
- [29] L. Landau and E. Lifshitz, *Theory of Elasticity* (Pergamon, New York, 1970).
- [30] B. Dubrovin, A. Fomenko, and S. Novikov, *Modern Geometry* (Springer, New York, 1984).
- [31] C. F. Li and Z. B. Hu, Phys. Rev. E **48**, 603 (1993).
- [32] T. Takigawa, T. Yamawaki, K. Takahashi and T. Masuda, Polym. Gels Networks **5**, 585 (1998).